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On periodically kicked oscillators

N D Sen Gupta
Tata Institute of Fundamental Research, Colaba, Mumbai-400 005, India

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Abstract : The time evolution of one and two dimensional oscillators which are kicked periodically with a potential is obtained The most interesting aspect of the investigation is that it leads to the existence of quasi-stationary states, which evolve independently of other similar states Further, if the potential has invariant subspaces, the system in its evolution is confined to these subspaces The method followed is the direct integration of the Schrödinger equation and then construct the wave function from the initial one.

Keywords : Schrodinger equation, kicked oscillators, quasi-stationary states

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1. Introduction

The quantum mechanical systems which are kicked at random or periodically have become recently a subject of intense study. The problem is an important one both from theoretical and observational point of view. Already an amount of literature has appeared [1–17], but most of these investigations are confined to one dimensional problems. In a recent paper [19iv], it has been pointed out that for systems which are kicked at random or periodically with a steady potential, there exists invariant (quasi-stationary) states. They are states which remain the same with the evolution of time, with only a change of phase at each kick. This property, which has been established qualitatively for general systems in the cited paper, is more evident in multi-dimensional systems.

The main object of the present paper is to show this explicitly for simple systems, namely one and two dimensional harmonic oscillators. Though periodic kicking is a special case of random kicking, yet periodic systems have their own important characteristics as far as the differential equation is concerned [18] and consequently the calculation becomes relatively simpler. This is due to fact that the corresponding Hill's determinant becomes considerably simple and reduces to a summation [19i]. This constitutes an additional attractive motivation for the presentation of the paper. Finally, the method followed here is direct in the sense that one can construct the solution of the Schrödinger equation from the initial condition. It may be mentioned that the investigation of the development of the system in time with the help of the evolution operator is, in general, begging the question. It presupposes the knowledge of the wave-function as the evolution operator is constructed from the wave function, which is to be obtained.

In the following section, the problem of one dimensional oscillator is studied in detail, pointing to the existence of quasi-stationary states. The next section is devoted to the two dimensional oscillator. The radial wave function for two dimensional case and its properties are given in the Appendix. The paper ends with a discussion.

2. One-dimensional periodically kicked oscillator

The Schrödinger equation of a harmonic oscillator of mass M and circular frequency ω, which is periodically kicked with period τ by a steady potential $\varepsilon f(x)$ is

$$\left\{ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} - \frac{M}{2}\omega^2 x^2 + \varepsilon f(x) \sum_{-\infty}^{\infty} \delta(t - n\tau) \right\} \Psi(x,t) = 0. \quad (1)$$

Writing x for $\sqrt{\frac{M\omega}{\hbar}}x$, one obtains

$$\left\{ i\frac{2}{\omega}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - x^2 + \frac{2\varepsilon}{\hbar\,\omega} f(x) \sum_{-\infty}^{\infty} \delta(t - n\tau) \right\} \Psi(x,t) = 0. \quad (2)$$

Since the coefficient of ε is periodic, $\Psi(x,t)$ can be written as (Floquet's Theorem),

$$\Psi(x,t) = \sum_{-\infty}^{\infty} \exp - i\left(\lambda + \frac{2\pi n}{\tau}\right) t \cdot \Psi_n(x) \quad (3)$$

(λ is a constant). Next, it can be shown that [19i]

$$\sum_{-\infty}^{\infty} \delta(t - n\tau) = \frac{1}{\tau}\sum_{-\infty}^{\infty} \exp. i2\pi m t/\tau \quad (4)$$

and

$$\sum_{-\infty}^{\infty} \delta(t - n\tau)\Psi(x,t) = \frac{1}{\tau}\sum_{-\infty}^{\infty}\left(\sum_{-\infty}^{\infty} \psi_q(x)\right) \exp. - i\left(\lambda + \frac{2\pi m t}{\tau}\right), \quad (5)$$

where q, m are integers. The orthonormal wave functions for the unkicked oscillator ($\varepsilon = 0$) are given by

$$\phi_\beta(x) = \frac{1}{\sqrt{2^\beta \beta! \sqrt{\pi}}} e^{x^2/2} \left(\frac{d}{dx}\right)^\beta e^{-x^2}, \quad (6)$$

β's are positive integer and $\phi_\beta(x)$ satisfies the equation

$$\left(\frac{d^2}{dx^2} - x^2 + 2\beta + 1\right)\phi_\beta(x) = 0. \tag{7}$$

Since $\phi_\beta(x)$'s form an orthonormal complete set, $\psi_q(x)$ can be expressed as

$$\psi_q(x) = \sum_0^\infty A_{q\beta}\phi_\beta(x), \tag{8}$$

$A_{q\beta}$ are constants. The first subscript q, in Roman, is the index for the Fourier component $(-\infty < q < \infty)$ pertaining to time variation and the second subscript β, in Greek, is the index related to the spatial eigen function $(0 \le \beta < \infty)$. Finally, substituting the expression for $\psi(x, t)$ from eqs. (3, 5, 8) in eq. (2) and equating the coefficients of exp. $i2\pi qt/\tau$ and $\phi_\gamma(x)$, one gets

$$\left\{\lambda + 2\pi\left(\frac{q}{\tau} - \frac{\gamma+1/2}{\tau_0}\right)\right\}A_{q\gamma} + \frac{\varepsilon}{\hbar\tau}\sum_0^\infty F_{\gamma\beta}a_\beta = 0, \tag{9}$$

where $$a_\beta = \sum_{-\infty}^\infty A_{s\beta}, \tau_0 = \frac{2\pi}{\omega}, \tag{10}$$

and $$F_{\gamma\alpha} = \int_{-\infty}^\infty \phi_\gamma(x)f(x)\phi_\alpha(x)dx, \tag{11}$$

Thus, $$a_\gamma + \frac{\varepsilon}{2\hbar}\left(\sum_{-\infty}^\infty \frac{1}{\Lambda_\gamma + \pi q}\right)\sum_0^\infty F_{\gamma\alpha}a_\alpha = 0, \tag{12}$$

where $$\Lambda_\gamma = \frac{\tau\lambda}{2} - \pi(2\gamma+1)\frac{\tau}{\tau_0}. \tag{13}$$

Hence, $$a_\gamma + \frac{\varepsilon}{2\hbar}\cot\Lambda_\gamma\sum_0^\infty F_{\gamma\alpha}a_\alpha = 0. \tag{14}$$

(a) Determination of λ:

The above relations show that Hill's determinant is reduced.and finally the corresponding characteristic indices λ are to be determined by solving eigen-value problem as stated below, $(|\lambda| < \frac{2\pi}{\tau}$, if λ^0 is a solution, $\lambda^0 + \frac{2\pi q}{\tau}$ (q an integer) is also a solution; note eq. [3]).

The eq. (14) can be written as

$$\iota'_\gamma + \frac{\varepsilon}{2\hbar}\sum_0^\infty F'_{\gamma\alpha}a'_\alpha = 0, \tag{15}$$

where $$a'_\gamma = \sqrt{\tan \Lambda_\gamma}\,.a_\gamma \tag{15'}$$

and $$F'_{\gamma\alpha} = \sqrt{\cot \Lambda_\gamma}\, F_{\gamma\alpha} \sqrt{\cot \Lambda_\alpha}\,. \tag{15''}$$

$F'_{\gamma\alpha}$ is real and symmetric, hence it can be expressed in terms of its orthonormal set of eigen-vectors u^θ and the corresponding eigen-values μ^θ,

$$F'_{\gamma\alpha} = \sum_0^\infty \mu^\theta u^\theta_\gamma \tilde{u}^\theta_\alpha\,. \tag{16}$$

Substituting this expression for $F'_{\gamma\alpha}$ in eq. (15) one obtains

$$\left(1 + \frac{\varepsilon}{2\hbar}\mu^\theta\right)\sum_0^\infty \left(\tilde{u}^\theta_\beta \,, a_\beta\right) = 0 \tag{17}$$

for $\theta = 0, 1, 2, \cdots$. Thus, for non-trivial solution

$$1 + \frac{\varepsilon}{2\hbar}\mu^\theta = 0 \tag{18}$$

which determines λ^θ, since μ^θ depends on λ^θ [eqs. (13) and (15)], to each u^θ and μ^θ, there corresponds a λ^θ.

(b) The Wave function :

From the expression for $\Psi(x, t)$ in eq. (3), one gets

$$\Psi(x,0) = \sum_0^\infty \left(\sum_{-\infty}^\infty A_{s\alpha}\right)\phi_\alpha(x) = \sum_0^\infty a_\alpha \phi_\alpha(x), \tag{19}$$

where $$a_\beta = \int_{-\infty}^\infty \phi_\beta(x)\Psi(x,0)dx. \tag{20}$$

Thus, a_γ is given by the initial condition. Having determined λ^β, the corresponding $\sum_0^\infty A_{q\gamma} u^\theta_\gamma$ can be obtained from eqs. (9, 20).

$$\sum_0^\infty A_{q\gamma} u^\theta_\gamma = -\frac{\varepsilon}{\hbar}\left\{\lambda\tau + 2\pi\left(q - \overline{\gamma + 1/2}\frac{\tau}{\tau_0}\right)\right\}^{-1} \sum_{\gamma,\beta=0}^\infty u^\theta_\gamma F_{\gamma\beta} a_\beta \tag{21}$$

and $$A_{q\gamma} = \sum_0^\infty u^\theta_\gamma A^\theta_q \,;\; A^\theta_q \equiv \sum_0^\infty A_{q\beta} u^\theta_\beta\,. \tag{22}$$

Thus, the wave function is completely determined from the initial condition. It is given by

$$\Psi(x,t) = \sum_0^\infty \left(\sum_0^\infty u^\theta_\gamma \phi_\gamma(x)\right)\sum_{-\infty}^\infty A^\theta_q \exp. - i\left(\lambda^\theta + 2\pi\frac{q}{\tau}\right)t. \tag{23}$$

The wave function evolves according to different time-dependent factors with its characteristic index λ^{θ} [19]. It can be easily checked that the normalization of the wave function persists with the evolution of time.

(c) Quasi-stationary states :

The general wave function [eq. (23)] with any arbitrary initial state $\Psi(x, 0)$, consists of distinct sets of oscillatory factor exp. $-i\lambda^{\theta}t$ apart from the periodic factor of period τ [eq (5)] [Note μ^{θ}, hence λ^{θ} are real, eq. (17)]. Further, we get from eq. (20)

$$\Psi(x,0) = \sum_{0}^{\infty} P^{\theta}\psi^{\theta}(x);\ \psi^{\theta}(x) = \sum_{\gamma}\tilde{u}_{\gamma}^{\theta}\phi_{\gamma}(x), \tag{24}$$

$P^{\theta} = \sum_{0}^{\infty} a_{\gamma}u_{\gamma}^{\theta}$ are constants. From eqs. (23) and (24), it is quite clear that any individual state $\psi^{\theta}(x)$ evolves completely independently of other $\psi^{\eta}(x,0) = \sum_{0}^{\infty} u_{\gamma}^{\eta}\phi_{\gamma}(x)$ and there is no interference or mixing among them. In spite of the kicks, there is no dispersion of these states in the usual sense of the term. It is all the more evident from eqs. (23) and (24) that if the initial state is any of the $\psi^{\theta}(x)$, it remains so through out its evolution in time These states may be considered as quasi-stationary states.

The time evolution of an unperturbed stationary state with energy $E_{\gamma} = \hbar\omega(\gamma + 1/2)$ is $\phi_{\gamma}\exp\ (-\frac{i}{\hbar}E_{\gamma}t) = \phi_{\gamma}\ \exp(-i\omega(\gamma + 1/2)$, the probability density $|\phi_{\alpha}|^2$ associated with it is constant in time. But for the quasi-stationary state, the time evolution is given by eq. (23)

$$\psi^{\theta}(x,t) = \sum_{0}^{\infty} u_{\gamma}^{0}\phi_{\gamma}(x)e^{-i\omega(\gamma+1/2)t}\sum_{-\infty}^{\infty} A_{q\gamma}^{\theta}\ \exp.-i\left(\lambda^{\theta} + \frac{2\pi q}{\tau}\right)t \tag{25}$$

and the probability associated with this state is

$$\left|\psi^{\theta}(x,t)\right|^2 = \sum_{0}^{\infty}\left|u_{\gamma}^{\theta}\right|^2\sum_{-\infty}^{\infty}\left|A_{q\gamma}^{\theta}\right|^2 + \text{oscillatory terms.} \tag{26}$$

The time average over the period τ of the oscillating term is zero, so that the average probability $\overline{|\psi^{\theta}(x,t)|^2}$ is constant similar to the case of stationary states.

(d) Special case :

Let us consider the special case when $\varepsilon f(x) \equiv \varepsilon$ (say) is a constant, since $F_{\gamma\beta} = \delta_{\gamma\beta}$, eq. (14) leads to

$$\left(1 + \frac{\varepsilon}{2\hbar}\cot A_{\gamma}\right)a_{\gamma} = 0. \tag{27}$$

$$\Lambda_\gamma = \frac{\tau}{2}\{\lambda_\gamma - \omega(\gamma+1/2)\} = -\cot^{-1}\frac{2\hbar}{\varepsilon} \equiv K(\text{say}), \tag{28}$$

$$\lambda_\gamma = \omega(\gamma+1/2) - \frac{2K}{\tau}, \tag{28'}$$

where K is a number. From eq. (9)

$$A_{q\gamma} = -\frac{\varepsilon}{2\hbar}\frac{1}{\lambda_\gamma + \dfrac{2\pi q}{\tau} - \omega(\gamma+1/2)}A_q. \tag{29}$$

Finally, in this case

$$\Psi(x,t) = \sum_{-\infty}^{\infty}\left(\sum_{0}^{\infty}\phi_\gamma A_{q\gamma}e^{-i\omega(\gamma+1/2)t}\right)\exp\left(+i(2K-2\pi q)\frac{t}{\tau}\right). \tag{30}$$

In fact, it can be checked that each individual term with $\phi_\gamma(x)$ is a solution. Thus, if initially the system is at one of the unperturbed stationary state, it remains in that state even with the kicks, which only introduce an ocillatory factor of period τ. Thus

$$\Psi^\beta(x,t) = \sum_{-\infty}^{\infty}\phi_\beta(x)e^{-i\omega(\gamma+1/2)t}A_{q\beta}\exp\left(i(2K-2\pi q)\frac{t}{\tau}\right), \tag{31}$$

with the initial $\phi_\beta(x)\exp(-i\omega(\beta+1/2)t)$, the only change is the additional oscillatory factor. This is also expected as x-independent potential do not produce any force The only change is in the phase factor at each kick. Consequently, for initially mixed stationary states, there is no mutual interference and each of the unperturbed stationary state evolve independently. These individual (unperturbed stationary) states are quasi-stationary states as the time average over a period $\overline{\tau|\Psi_\beta(x,t)|^2}$ remains the same as $|\psi_\beta(x,t)|^2$, the unperturbed state.

3. Two-dimensional oscillator

The wave equation for a two-dimensional periodically kicked oscillator is given by

$$\left\{i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right) - \frac{M\omega^2}{2}(x^2+y^2)\right.$$
$$\left. + \varepsilon F(x,y)\sum_{-\infty}^{\infty}\delta(t-n\tau)\right\}\Psi(x,y;t) = 0. \tag{32}$$

Introducing the dimension-less variables as before, the wave equation for the steady unperturbed oscillator may be written as

$$\left\{\left(\frac{d^2}{dx^2}-x^2\right) + \left(\frac{d^2}{dy^2}-y^2\right) + E'\right\}\Psi^0(x,y) = 0. \tag{33}$$

The wave-function is given by

$$\Psi_N^0(x,y) = \text{const.}\sum_{\alpha,\beta}\phi_\beta(x)\phi_\gamma(y)B_{\beta\gamma} \tag{34}$$

and $$E' = 2(\beta+\gamma)+2 = N(\text{say}).$$

The states are degenerate for a given energy *i.e.* fixed N, there are $N+1$ states. Since the potential $F(x, y)$ [eq. (32)] which are of physical interest, depends on $\rho = \sqrt{x^2+y^2}$, it is imperative to introduce plane polar coordinates (ρ, θ). The wave equation is now given by

$$\left\{\frac{i2}{\omega}\frac{\partial}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} - \rho^2 \right.$$
$$\left. + \frac{2\varepsilon}{\hbar\omega}F(\rho)\sum_{-\infty}^{\infty}\delta(t-\eta\tau)\right\}\Psi(\rho,\theta;t) = 0. \tag{35}$$

The wave fuctions for the unperturbed part of this equation is treated in detail in the Appendix. The steady unkicked wave function is given by $\Phi_\eta^m(\rho)$ exp. $im\theta$, m is any integer and η is a positive integer. $\Phi_\eta^m(\rho)$ satisfies eq. (A3), for energy $E = 2\eta + m + 1$. Since $\Phi_\eta^m(\rho)$'s form a complete set, and the coefficient of eq. (35) is periodic, the wave function $\Psi_\eta^m(\rho,\theta;t)$ can be written as

$$\Psi(\rho,\theta;t) = \sum_{q,m=-\infty}^{\infty}\sum_{\eta=0}^{\infty}A_{q\eta}^m\Phi_\eta^m(\rho)e^{im\theta}\exp. -i\left(\lambda+2\pi\frac{q}{\tau}\right)t, \tag{36}$$

where $A_{q\eta}^m$'s are constants [19]. Substituting this expression for $\Psi(\rho, \theta; t)$ in eq. (35) and proceeding as before (taking note of eqs. (4–5)), one gets

$$\left[\lambda + 2\pi\left\{\frac{q}{\tau} - \frac{1}{\tau_0}\left(\eta+\frac{m+1}{2}\right)\right\}\right]A_{q\eta}^m = -\frac{\varepsilon}{\hbar\tau}\sum_0^\infty F_{\eta\zeta}^{mm}a_\zeta^m. \tag{37}$$

where $$F_{\eta\zeta}^{mn} = \int_0^\infty\int_0^{2\pi}\Phi_\eta^m(\rho)F(\rho)\Phi_\zeta^n e^{i(n-m)\theta}\rho d\theta d\rho \tag{38}$$

and $$a_\zeta^m = \sum_{-\infty}^{\infty}A_{q\zeta}^m = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\phi_\gamma^m(\rho)e^{im\theta}\Psi(\rho,\theta;0)\rho d\theta d\rho. \tag{39}$$

$$F_{\eta\zeta}^{mn} = 0,\ m \neq n; \tag{40}$$

$$a_\eta^m + \frac{\varepsilon}{2\hbar}\cot\Lambda'_\eta\sum_0^\infty F_{\eta\zeta}^{mm}a_\zeta^m = 0, \tag{41}$$

where $$\Lambda'_\eta = \frac{\lambda_\eta\tau}{2} - \pi(2\eta+m+1)\frac{\tau}{\tau_0}. \tag{42}$$

Now, one can proceed exactly as before for the one-dimensional case (eqs. (15)). Let v^{β} and v^{β} be the eigen vectors and eigen values ($\theta = 1, 2, \ldots$) of $\sqrt{\cot \Lambda'_{\eta}}\, F_{\eta\zeta}$, leading to

$$\Lambda'_{\beta} + \frac{\varepsilon}{2\hbar} v^{\beta} = 0, \tag{43}$$

similar to eqs. (17) and (18). For each eigen value v^{β}'s, one obtains a Λ'_{β}; with this value of Λ'_{β}, one gets $A^{m}_{q\beta}$ from eq. (37), which determines the wave function completely from eq. (36)

(a) Independence of angular momentum states :

In characterizing the nature of the quasi-stationary states in this case, one first notes that $A^{m}_{q\zeta}$ are determined from a^{m}_{ζ} only, as a consequence of eq. (40). This is due to the fact that the potential $F(\rho)$ depends only on ρ, and the corresponding matrix has invariant subspaces with fixed angular momentum states $(-i\hbar \frac{\partial}{\partial\theta} \equiv m)$. The states with given angular momenta evolve among themselves and there is no dispersion in the usual sense of the term. This restricted nature of evolution has been shown in general, with the potential function having invariant subspace [19iv].

(b) Quasi-stationary states :

Next, the general wave function $\Psi(\rho, \theta; t)$ with arbitrary initial state $\Psi(\rho, \theta; 0)$ given by eq (26), consists of distinct sets of oscillatory factors $\exp(-iv^{\beta}t)$, apart from the periodic factor of period τ. Further,

$$\Psi_{m}(\rho,\theta;0) = \sum_{0}^{-} Q^{\beta}_{m} \Psi^{\beta}_{m}(\rho,\theta;0); Q^{\beta}_{m} = \sum_{0}^{-} a^{m}_{\gamma} v^{\beta}_{\gamma},$$

$$\Psi^{\beta}_{m}(\rho,\theta;0) = \sum v^{\beta}_{\gamma} \phi^{m}_{\gamma}(\rho)\, e^{im\theta} \tag{44}$$

Q^{β}_{m}'s are constants. From eqs. (36) and (41), it is clear that the state $\Psi^{p}_{m}(\rho,\theta;t)$ evolves from $\Psi^{\beta}_{m}(\rho,\theta;0)$ completely independently and there is no mixing among other $\Psi^{\alpha}_{m}(\phi,\theta;t)$ with $\alpha \neq \beta$. Further, if the initial state happens to be one of this state, then the system remains in the same state after the kicks throughout its evolution.

4. Discussion

Though the investigation in the paper is confined to harmonic oscillator, the method followed here may be carried over *mutatis mutandis* to any finite dimensional system with discrete spectrum. The essential point to be emphasized is that even under kicking, there are states which evolve independently without interfering with other states. These states are specified by the potential. They are the quasi-stationary states, the probabilities associated with them averaged over a period are constant. Even in one dimension, there exists these quasi-stationary states as shown in Section 2(d) and it is too much to say that the perturbation induces loss of memory.

In the previous paper [19iv], it was shown qualitatively that the effect of kicking (be it periodic or random) is to impart discontinuous phase factor to the stationary wave function of the unkicked system. The wave function obtained here appears to be continuous (eqs. (23) and (36)). But this is apparent as each of them contains a series

$$\sum_{-\infty}^{\infty} \frac{\exp(i\pi(p+k)t/\tau)}{p+k}$$

This diverges at $t = n\tau$, as it should be due to δ-function and its derivative is a δ-function This leads to a discontinuous change at each $n\tau$.

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Appendix

The equation for the steady unperturbed wave function $\psi(\rho, \phi)$ is given by

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} - \rho^2 + 2E\right)\psi(\rho,\theta) = 0. \tag{A1}$$

Hence,
$$\psi(\rho,\theta) = \sum_{-\infty}^{\infty} \psi^m(\rho)\exp.im\theta, \tag{A2}$$

where m is an integer and

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} - \rho^2 + 2E\right)\psi^m(\rho) = 0. \tag{A3}$$

It can be easily shown that it has bounded solution (*i.e.*, $\psi^m(\rho \to \infty) = 0$) only when

$$E = 2\eta + m + 1, \tag{A4}$$

where η is a positive integer. Here and in the sequel m stands for $|m|$. The radial part of the wave function is given by

$$\Phi^m_\eta(\rho) = e^{-\rho^2/2}\rho^m\sum_0^n a_p\rho^{2p} \tag{A6}$$

$$a_{p+1} = \frac{p-\eta}{2p(p+m)}a_p. \tag{A6'}$$

$e^{\rho^2/2}\Phi^m_\eta(\rho)$ is a polynomial of degree $2\eta + m$. This expression can be obtained directly from eq. (A3) which leads to a representation of $\Phi^m_n(\rho)$ similar to the Hermite polynomials. The recurrence relations between them needed for evaluating the matrix elements of $F(\rho)$ in eq. (38), can be obtained easily.

From eqs. (34) and (A3), the most general eigen function corresponding to

$$E = \alpha + \beta + 1 = 2\eta + m + 1 \tag{A7}$$

is the linear combination

$$\sum B_{\alpha\beta}\phi_\alpha(x)\phi_\beta(y) : \alpha + \beta = 2\eta + m. \tag{A7'}$$

It contains various terms with factors as exp $(ip\theta)$, $-(2\eta + m) \le p \le 2\eta + m$ but $\psi^m(\rho)$ is associated with factor exp $(im\theta)$. This can be picked up very easily by expressing them in terms of u, v as given by,

$$2u = x + iy,\ 2v = x - iy \tag{A8}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v};\ \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right); \tag{A8'}$$

and $$\frac{\partial}{\partial u} \equiv e^{i\theta}\left(\frac{\partial}{\partial \rho} + \frac{i}{\rho}\frac{\partial}{\partial \theta}\right);\ \frac{\partial}{\partial v} = \left(\frac{\partial}{\partial u}\right)^{*} \tag{A8''}$$

A prototype term of the expression (A7'), leaving aside constant factors and summation, is with the help of eq. (6) for $\phi_\alpha(x)$ and $\phi_\beta(y)$,

$$\phi_\alpha(x)\phi_\beta(y) \simeq e^{\rho^2/2}\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta} e^{-\rho^2} \tag{A9}$$

use $$\simeq e^{\rho^2/2}\left(\frac{\partial}{\partial u}\right)^{\sigma}\left(\frac{\partial}{\partial v}\right)^{\nu} e^{-\rho^2} \text{ (from (A8''))} \tag{A9'}$$

where $$\sigma + \nu = \alpha + \beta. \tag{A9''}$$

It can be easily shown (with the help A8'') by induction that

$$\left(\frac{\partial}{\partial v}\right)^{\nu} e^{-\rho^2} = e^{-i\nu\theta}\rho^{\nu}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\nu} e^{-\rho^2} = e^{-i\nu\theta}\rho^{\nu}e^{-\rho^2}(-2)^{\nu} \tag{A10}$$

and $$\left(\frac{\partial}{\partial u}\right)^{\sigma} e^{-i\nu\theta}\rho^{\nu}e^{-\rho^2} = e^{i(\sigma-\nu)\theta}\rho^{\sigma-\nu}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\sigma}\rho^{2\nu}e^{-\rho^2} \tag{A10'}$$

Here, only θ-dependent term is $e^{i(\sigma-\nu)\theta}$, hence to obtain the terms with $e^{im\theta}$ and $\psi^m_\eta(\rho)$ in (A7), one must have

$$\sigma - \nu = m \text{ and } \eta = \nu. \tag{A10''}$$

Hence, $$\Phi^m_\eta(\rho) = e^{\rho^2/2}\rho^m\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\eta+m}\rho^{2\eta}e^{-\rho^2} \tag{A11}$$

$$= e^{-\rho^2/2}R^m_\eta(\rho), \tag{A12}$$

where the polynomial

$$R^m_\eta(\rho) = \rho^m e^{\rho^2}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\eta+m}\rho^{2\eta}e^{-\rho^2} \tag{A13}$$

and $$R^m_\eta(\rho) = \rho^m\left(\frac{d}{\rho d\rho}\right)^{m} R^0_\eta(\rho). \tag{A14}$$

One finds directly from eq. (A3) and by partial integration that

$$\int_0^\infty \rho\Phi^m_\eta(\rho)\Phi^m_\zeta(\rho)d\rho = (-1)^\eta 2^{\eta+m-1}\eta!\delta_{\eta\zeta}. \tag{A15}$$

Thus $\sqrt{\rho}\phi^m_\eta$'s are mutually orthogonal functions. It can be shown that they form a complete set, so that any function with same boundary condition, namely bounded at $\rho = 0$ and tends to zero as $\rho \to \infty$ can be expressed in terms of $\Phi^m_\eta(\rho)$; (m fixed and $\eta = 0, 1, 2, \ldots$).

The recurrence relation between them may be obtained by writing

$$\rho^2 R^m_\eta(\rho) = \rho^m \rho^2 \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{\eta+m} \rho^{2\eta}\left(-\frac{1}{2\rho}\frac{d}{d\rho}\right)e^{-\rho^2} \tag{A16}$$

and shifting ρ^2 to the right and $\frac{1}{\rho}\frac{d}{d\rho}$ to the left,

$$R^m_{\eta+1} + 4\eta(\eta+m)R^m_{\eta-1} = 2(2\eta+m+1-\rho^2)R^m_\eta . \tag{A17}$$